Families of strictly isospectral potentials

1989 J. Phys. A: Math. Gen. 22 L987

(http://iopscience.iop.org/0305-4470/22/21/002)

J. Phys. A: Math. Gen. **22** (1989) L987-L992. Printed in the UK

LETTER TO THE EDITOR

Families of strictly isospectral potentials

Wai-Yee Keung†, Uday P Sukhatme†, Qinmou Wang†§ and Tom D Imbo‡

† Department of Physics, University of Illinois at Chicago, IL 60680, USA
‡ Center for Particle Theory, University of Texas at Austin, TX 78712, USA

Received 31 July 1989

Abstract. For any potential $V(x)$ which holds n bound states, we use repeated supersymmetry transformations to construct an n-parameter family of 'strictly isospectral' potentials (identical eigenenergies, reflection and transmission amplitudes for all family members). We investigate how the shape and behaviour of various isospectral potentials changes as the n parameters are varied. Contained in this family are potentials with n widely separated wells, each of which holds a single bound state.

Recently there have been many applications of supersymmetry (SUSY) [1] in quantum mechanics. New insights concerning the solutions of the one-dimensional time-independent Schrödinger equation have been obtained in areas such as solvable models [2], large-N expansions [3], WKB approximations [4], tunnelling [5] and S-matrix computations [6]. The supersymmetric approach used in the above applications essentially corresponds to the Darboux transformation [7] in the theory of second-order linear differential equations. The key step is the construction of a pair of supersymmetric partner potentials with the same energy eigenvalues except for the ground state. An interesting physical problem is to determine the most general form of a potential with specified energy eigenvalues and scattering matrix. It has been shown how supersymmetry allows one to construct a one-parameter family of potentials $\hat{V}_0(\lambda; x)$ strictly isospectral to an input potential $V_0(x)$ [8-11]. In this letter, we generalise this idea and construct a much larger n-parameter family by repeated supersymmetry operations, where n is the number of bound states in the initial potential $V_0(x)$. The generated family contains potentials which are strictly isospectral (they all have the same eigenvalues, reflection and transmission amplitudes) [11]. They are intimately connected with solutions of non-linear partial differential equations such as the KdV equation [12-14]. Similar results have also been obtained using the inverse scattering approach (for a review, see [15]). However, our approach is different and our final explicit formula for the n-parameter family of potentials is much simpler than the corresponding formula involving an $n \times n$ determinant obtained by solving the Gel'fand-Levitan-Marchenko integral equation by various methods [15, 16].

Starting with a potential $V_0(x)$ with the ground state ψ_0 at energy E_0, we first generate the SUSY partner potential V_1 which is *almost* isospectral to V_0. V_1 has the same eigenvalues as V_0 except that it has no bound state at E_0. We have

$$V_1 = V_0 - 2\frac{d^2}{dx^2}\ln \psi_0.$$

§ Permanent address: Anhui Normal University, People's Republic of China.

(The convention $\hbar = 2m = 1$ is used throughout the letter.) This is the standard SUSY procedure for deleting the ground state of a potential. The reflection and transmission amplitudes of V_0 and V_1 are, of course, different but simply related [11]. The ground state ψ_1 for the potential V_1 is located at energy E_1. The procedure can be repeated 'upward', producing potentials $V_2, V_3, \ldots$, with ground states $\psi_2, \psi_3, \ldots$, at energies $E_2, E_3, \ldots$, until the top potential $V_n(x)$ holds no bound state (see figure 1, which corresponds to $n = 2$). Although the potential V_1 does not have an eigenenergy E_0, the function $1/\psi_0$ satisfies the Schrödinger equation with potential V_1 and energy E_0. The other linearly independent solution is $\int_{-\infty}^{x} \psi_0^2(x')\,\mathrm{d}x'/\psi_0$. Therefore, the most general solution of the Schrödinger equation for the potential V_1 at energy E_0 is

$$\Phi_0(\lambda_0) = (\mathscr{I}_0 + \lambda_0)/\psi_0$$

where

$$\mathscr{I}_i \equiv \int_{-\infty}^{x} \psi_i^2(x')\,\mathrm{d}x'.$$

Now, starting with a potential V_1, we can again use the standard SUSY procedure [9] to add a state at E_0 by using the general solution $\Phi_0(\lambda_0)$

$$\hat{V}_0(\lambda_0) = V_1 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln \Phi_0(\lambda_0).$$

The function $1/\Phi_0(\lambda_0)$ is the normalisable ground-state wavefunction of $\hat{V}_0(\lambda_0)$, provided [11] that λ_0 does not lie in the interval $-1 \leqslant \lambda_0 \leqslant 0$. Therefore, we find a one-parameter family of potential $\hat{V}_0(\lambda_0)$ isospectral to V_0 for $\lambda_0 > 0$ or $\lambda_0 < -1$:

$$\hat{V}_0(\lambda_0) = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln(\psi_0\Phi_0(\lambda_0))$$
$$= V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln(\mathscr{I}_0 + \lambda_0).$$

Note that this family contains the original potential V_0. This corresponds to the choices $\lambda_0 \to \pm\infty$. Also, V_0 and $\hat{V}_0(\lambda_0)$ have the same reflection and transmission amplitudes, that is $\hat{R}(k) = R(k)$, $\hat{T}(k) = T(k)$ [11].

In order to produce a two-parameter family of isospectral potentials, we go from V_0 to V_1 to V_2 by successively deleting the two lowest states of V_0 and then we re-add

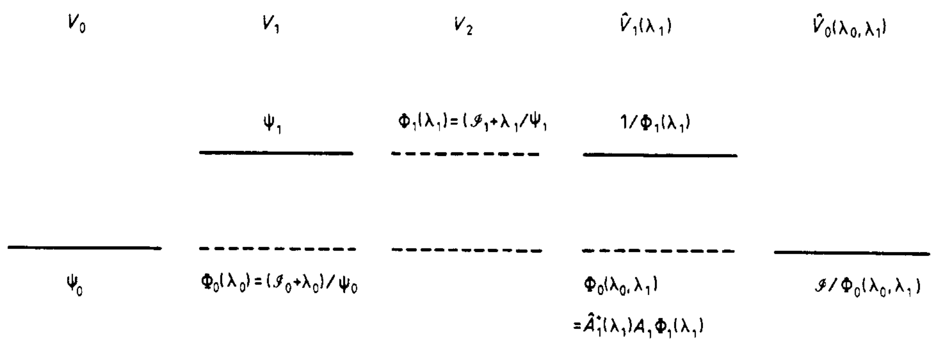

Figure 1. Schematic diagram showing the SUSY transformation for deleting two states and re-adding them, thus producing a two-parameter (λ_0, λ_1) isospectral family.

the two states at E_1 and E_0 by SUSY transformations. The most general solutions of the Schrödinger equation for the potential V_2 are given by $\Phi_1(\lambda_1)=(\mathscr{I}_1+\lambda_1)/\psi_1$ at energy E_1, and $A_1\Phi_0(\lambda_0)$ at energy E_0 (see figure 1). Here the SUSY operator A_i relates solutions of V_i and V_{i+1}

$$A_i=\frac{d}{dx}-(\ln\psi_i)'.$$

The prime denotes differentiation with respect to x. Then, as before, we find an isospectral one-parameter family $\hat{V}_1(\lambda_1)$

$$\hat{V}_1(\lambda_1)=V_1-2\frac{d^2}{dx^2}\ln(\mathscr{I}_1+\lambda_1).$$

The solutions of the Schrödinger equation for potentials V_2 and $\hat{V}_1(\lambda_1)$ are related by a new SUSY operator

$$\hat{A}_1^{\dagger}(\lambda_1)=-\frac{d}{dx}+(\ln\Phi_1(\lambda_1))'.$$

Therefore, the solution $\Phi_0(\lambda_0,\lambda_1)$ at E_0 for $\hat{V}_1(\lambda_1)$ is

$$\Phi_0(\lambda_0,\lambda_1)=\hat{A}_1^{\dagger}(\lambda_1)A_1\Phi_0(\lambda_0).$$

The normalisable function $1/\Phi_0(\lambda_0,\lambda_1)$ is the ground state at E_0 of a new potential, which results in a two-parameter family of isospectral systems $\hat{V}_0(\lambda_0,\lambda_1)$

$$\begin{aligned}\hat{V}_0(\lambda_0,\lambda_1)&=V_0-2\frac{d^2}{dx^2}\ln(\psi_0\psi_1\Phi_1(\lambda_1)\Phi_0(\lambda_0,\lambda_1))\\&=V_0-2\frac{d^2}{dx^2}\ln(\psi_0(\mathscr{I}_1+\lambda_1)\Phi_0(\lambda_0,\lambda_1))\end{aligned}$$

for $\lambda_i>0$ or $\lambda_i<-1$. A useful alternative expression is

$$\hat{V}_0(\lambda_0,\lambda_1)=-\hat{V}_1(\lambda_1)+2(\Phi_0'(\lambda_0,\lambda_1)/\Phi_0(\lambda_0,\lambda_1))^2+2E_0.$$

The above procedure is best illustrated by the pyramid structure in figure 1. It can be generalised to an n-parameter family of isospectral potentials for an initial system with n bound states. The formulae for an n-parameter family are

$$\Phi_i(\lambda_i)=(\mathscr{I}_i+\lambda_i)/\psi_i \qquad i=0,\ldots,n-1$$

$$A_i=\frac{d}{dx}+(\ln\psi_i)'$$

$$\hat{A}_i^{\dagger}(\lambda_i,\ldots,\lambda_{n-1})=-\frac{d}{dx}+(\ln\Phi_i(\lambda_i,\ldots,\lambda_{n-1}))'$$

$$\begin{aligned}\Phi_i(\lambda_i,\lambda_{i+1},\ldots,\lambda_{n-1})\\&=\hat{A}_{i+1}^{\dagger}(\lambda_{i+1},\lambda_{i+2},\ldots,\lambda_{n-1})\hat{A}_{i+2}^{\dagger}(\lambda_{i+2},\lambda_{i+3},\ldots,\lambda_{n-1})\ldots\hat{A}_{n-1}^{\dagger}(\lambda_{n-1})\\&\quad\times A_{n-1}A_{n-2}\ldots A_{i+1}\Phi_i(\lambda_i)\end{aligned}$$

$$\hat{V}_0(\lambda_0,\ldots,\lambda_{n-1})=V_0-2\frac{d^2}{dx^2}\ln(\psi_0\psi_1\ldots\psi_{n-1}\Phi_{n-1}(\lambda_{n-1})\ldots\Phi_0(\lambda_0,\ldots,\lambda_{n-1})).$$

The above equations summarise the main results of this letter. Note that although all the results obtained are on the full line ($-\infty < x < \infty$), they can be readily used for the half-line ($0 < r < \infty$) corresponding to central potentials in three dimensions. In fact, we will study the example of the Coulomb potential later.

As a first application, we consider potentials of the form

$$V_0 = -N(N+1)\,\mathrm{sech}^2\, x$$

where N is an integer. These potentials are of special physical interest since they are reflectionless as well as shape invariant [2, 17]. V_0 holds N bound states, and we may form an N-parameter family of isospectral potentials. We start with the simplest case $N=1$. We have $V_0 = -2\,\mathrm{sech}^2\, x$, $E_0 = -1$ and $\psi_0 = 2^{-1/2}\,\mathrm{sech}(x)$. The corresponding 1-parameter family is

$$\hat{V}_0(\lambda_0) = -2\,\mathrm{sech}^2(x + \tfrac{1}{2}\ln(1 + 1/\lambda_0)).$$

Clearly, varying the parameter λ_0 corresponds to translations of $V_0(x)$. As λ_0 approaches the limits 0^+ (Pursey limit [9]) and -1^- (Abraham-Moses limit [10]), the minimum of the potential moves to $-\infty$ and $+\infty$ respectively.

For the case $N=2$, $V_0 = -6\,\mathrm{sech}^2\, x$ and there are two bound states at $E_0 = -4$ and $E_1 = -1$. The SUSY partner potential is $V_1 = -2\,\mathrm{sech}^2\, x$. The ground-state wavefunctions of V_0 and V_1 are $\psi_0 = (\sqrt{3}/2)\,\mathrm{sech}^2\, x$ and $\psi_1 = (1/\sqrt{2})\,\mathrm{sech}(x)$. Also, $\mathscr{I}_0 = \frac{1}{4}(3\tanh x - \tanh^3 x + 2)$ and $\mathscr{I}_1 = \frac{1}{2}(\tanh x + 1)$. After some algebraic work, we obtain the two-parameter family

$$\hat{V}_0(\lambda_0, \lambda_1) = -12\,\frac{[3 + 4\cosh(2x - 2\delta_1) + \cosh(4x - 2\delta_0)]}{[\cosh(3x - \delta_1 - \delta_0) + 3\cosh(x + \delta_1 - \delta_0)]^2}$$

$$\delta_i = -\tfrac{1}{2}\ln(1 + 1/\lambda_i) \qquad i = 1, 2.$$

As we let $\lambda_0 \to -1$, a well with one bound state at E_0 will move in the x direction, leaving behind a shallow well with one bound state at E_1. The movement of the shallow well is essentially controlled by the parameter λ_1. Thus, we have the freedom to move either of the wells†.

For the case $N=3$, $V_0 = -12\,\mathrm{sech}^2\, x$ and the potential has three bound states at $E_0 = -9$, $E_1 = -4$ and $E_2 = -1$. Following the above algorithm, we compute a three-parameter isospectral family. In figure 2, we plot $\hat{V}_0(\lambda_0, \lambda_1, \lambda_2)$ for the choices $\lambda_0 = \lambda_1 = \lambda_2 = \lambda = \infty, -1.0001$. The potential for $\lambda = \infty$ is just the initial potential V_0, whereas for $\lambda = -1.0001$, we see three separated wells, each holding one bound state. If only a single parameter λ_i is taken to the limiting value -1 (keeping other parameters arbitrary), then only one well, having a bound state at energy E_i, moves to $x = \infty$. It can be shown [14] that this well has the form $V(x) = -2\alpha^2\,\mathrm{sech}^2(\alpha x)$, $\alpha \equiv \sqrt{-E_i}$. From the above discussion we can conclude that this separation into many wells is also true for general potentials.

As a second example we consider the Coulomb potential. The s-wave effective potential is $V_0(r) = -e^2/r$ where we choose $e^2 = 2$. Its SUSY partner is $V_1(r) = 2/r^2 - 2/r$. The ingredients for constructing a two-parameter family $\hat{V}_0(\lambda_0, \lambda_1)$ are $E_0 = -1$, $E_1 = -\frac{1}{4}$, $\psi_0 = 2r\,e^{-r}$, $\psi_1 = r^2\,e^{-r/2}/\sqrt{24}$, $\mathscr{I}_0 = 1 - e^{-2r}(1 + 2r + 2r^2)$, and $\mathscr{I}_1 = 1 - e^{-r}(1 + r + \frac{1}{2}r^2 + \frac{1}{6}r^3 + \frac{1}{24}r^4)$. We can construct the two-parameter family $\hat{V}_0(\lambda_0, \lambda_1)$ from this information. In figure 3, we have plotted some of the members of the two-parameter family. Keeping λ_0 fixed at a value -1.1, we have varied λ_1. The curves

† If one chooses $\delta_0 = 32t$ and $\delta_1 = 4t$, $\hat{V}$ is the well known two-soliton solution of the KdV equation. The potentials shown in figure 2 are related to the three-soliton solution. These issues are discussed in [14].

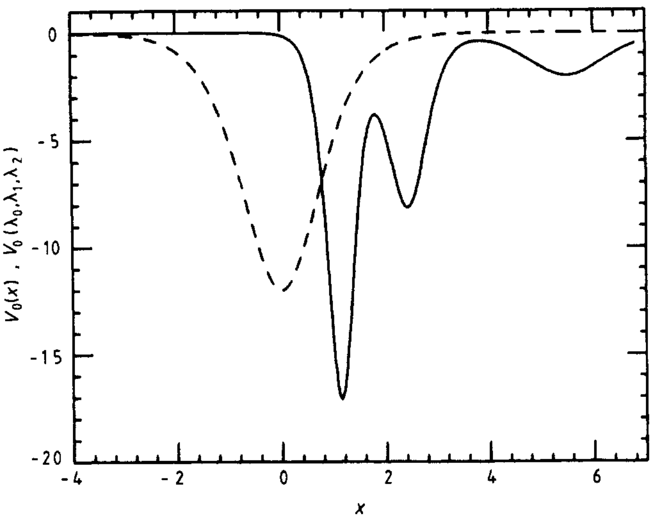

Figure 2. Isospectral three-parameter family $\hat{V}_0(\lambda_0, \lambda_1, \lambda_2)$ for the input potential $V(x) = -12\,\mathrm{sech}^2\,x$ (broken curve) which holds three bound states. We show the case $\lambda_0 = \lambda_1 = \lambda_2 = -1.0001$ (full curve).

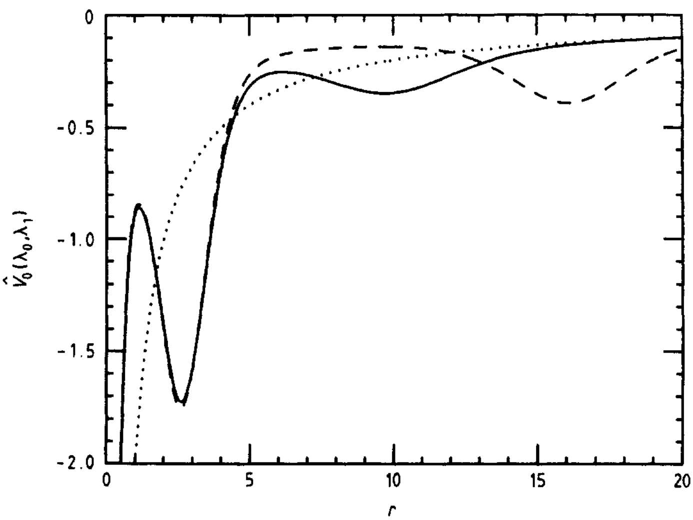

Figure 3. Isospectral two-parameter family for the input Coulomb potential $V(r) = -2/r$ (dotted curve). We show the cases $\lambda_1 = -1.1$ (full curve) and $\lambda_1 = -1.001$ (broken curve) for fixed $\lambda_0 = -1.1$.

correspond to $\lambda_1 = -1.1, -1.001$. These choices illustrate how the shallower well with bound state at $E_1 = -\frac{1}{4}$ moves to $r = \infty$ as $\lambda_1 \to -1$.

In conclusion, we have shown how to construct an n-parameter family of isospectral potentials for any given $V_0(x)$ using supersymmetry transformations†. If $V_0(x)$ is

† In this letter we have restricted our attention to adding and deleting states using SUSY. Two other closely related procedures are that of Pursey [9] and Abraham–Moses [10]. By combining these procedures, other distinct n-parameter isospectral families can be found, but they have different phase shifts. This point is discussed in [18].

exactly solvable (or quasi-exactly-solvable) [14], then our procedure yields an n-parameter family of new solvable potentials. These potentials can be useful starting points for perturbation theory calculations. We have also used them to construct explicit, pure multi-soliton solutions of the KdV and other non-linear evolution equations.

This research was supported by the US Department of Energy.

References

[1] Witten E 1981 *Nucl. Phys.* B **185** 513
Cooper F and Freedman B 1983 *Ann. Phys., NY* **146** 262
[2] Dutt R, Khare A and Sukhatme U 1988 *Am. J. Phys.* **56** 163
Dabrowska J, Khare A and Sukhatme U 1988 *J. Phys. A: Math. Gen.* **21** L195
[3] Imbo T and Sukhatme U 1985 *Phys. Rev. Lett.* **54** 2184
[4] Comtet A, Bandrauk A and Campbell D 1985 *Phys. Lett.* **105B** 159
Khare A 1985 *Phys. Lett.* **161B** 131
Dutt R, Khare A and Sukhatme U 1986 *Phys. Lett.* **181B** 295
[5] Bernstein M and Brown L S 1984 *Phys. Rev. Lett.* **52** 1933
Keung W-Y, Kovacs E and Sukhatme U 1988 *Phys. Rev. Lett.* **60** 41
[6] Amado R D 1988 *Phys. Rev.* A **37** 2277
Baye D 1987 *Phys. Rev. Lett.* **58** 2738
Cooper F, Ginocchio J N and Wipf A 1988 *Phys. Lett.* **129** 145
Khare A and Sukhatme U 1988 *J. Phys. A: Math. Gen.* **22** L501
Amado R D, Cannata F and Dedonder J P 1988 *Phys. Rev. Lett.* **61** 2901
[7] Darboux G 1882 *Comptes Rendus* **94** 1456
[8] Nieto M M 1984 *Phys. Lett.* **145B** 208
Sukumar C V 1985 *J. Phys. A: Math. Gen.* **18** 2917
Chaturvedi S and Raghunathan K 1986 *J. Phys. A: Math. Gen.* **19** L775
[9] Luban M and Pursey D L 1986 *Phys. Rev.* D **33** 431
Pursey D L 1986 *Phys. Rev.* D **33** 1048, 2267
[10] Abraham P B and Moses H E 1980 *Phys. Rev.* A **22** 1333
[11] Khare A and Sukhatme U 1989 *J. Phys. A: Math. Gen.* **22** 2847
[12] Kwong W and Rosner J L 1986 *Prog. Theor. Phys. Suppl.* **86** 366
[13] Lamb G L 1980 *Elements of Soliton Theory* (New York: Wiley)
[14] Wang Q, Sukhatme U, Keung W-Y and Imbo T 1989 *Preprint*
[15] Newton R G 1982 *Scattering Theory of Waves and Particles* (Berlin: Springer)
Chadan K and Sabatier P C 1977 *Inverse Problems in Quantum Scattering Theory* (Berlin: Springer)
[16] Deift P and Trubowitz E 1979 *Commun. Pure Appl. Math.* **32** 121
[17] Gendenshtein L E 1983 *JETP Lett.* **38** 356
Cooper F, Ginocchio J N and Khare A 1987 *Phys. Rev.* D **36** 2458
[18] Sukhatme U and Khare A 1989 *Preprint* University of Illinois at Chicago UICHEP/89-4
[19] Turbiner A V 1988 *Commun. Math. Phys.* **118** 467
Shifman M A 1989 *Mod. Phys.* A **4** 2897